

GENERIC FIBER OF POWER SERIES RING EXTENSIONS

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ABSTRACT. Let D be a Noetherian domain containing a field, $a \in D$ a nonzero nonunit and z an indeterminate over D . We prove that the generic fiber of the extension $D[[z]] \hookrightarrow D[1/a][[z]]$ has dimension $\geq \dim(D/aD)$.

2000 Mathematics Subject Classification: 13F25, 13J10.

Key words and phrases: trivial generic fiber, complete local ring, analytic independence.

1. Introduction.

An extension $A \subseteq B$ of integral domains is said to be a *trivial generic fiber* (TGF) extension, if every nonzero prime ideal of B has nonzero intersection with A . Let K be a field and z, x_1, x_2, \dots, x_n indeterminates over K .

In [4], Heinzer, Rotthaus and Wiegand proved that the mixed polynomial/power series ring extension

$$K[x_1, x_2, \dots, x_n][[z]] \hookrightarrow K[x_1, x_2, \dots, x_n, 1/x_1][[z]]$$

is TGF for $n = 1$ and non-TGF for $n = 3$. They asked what happens for $n = 2$.

In this note, we study the TGF property for power series extensions of type

$$D[[z]] \hookrightarrow D_a[[z]]$$

where D is a domain, $0 \neq a \in D$ and z an indeterminate over D . Here D_a is a (short) notation for fraction ring $D[1/a]$. We prove that $D[[z]] \hookrightarrow D_a[[z]]$ is TGF when D is one-dimensional.

Let A be a domain and B a ring between $A[x, y]$ and $A[[x, y]]$, where x, y, z are indeterminates over A . We show that the extension $B[[z]] \hookrightarrow B_x[[z]]$ is not TGF. In particular, for a field K , the extension

$$K[x, y][[z]] \hookrightarrow K[x, y]_x[[z]]$$

is not TGF, thus answering [4, Question 4.8] mentioned above. The key point of our proof is that, as shown by Abhyankar and Moh in [1, Theorem 2], there exist elements $\lambda \in {}_z K[x]_x[[z]]$ which are analytically independent over $K[[x, z]]$ (e.g., $x \sum_{n \geq 1} (z/x)^{n!}$). Then it is easy to find a prime ideal of $K[x, y]_x[[z]]$ containing $\lambda - y$ which has zero intersection with $K[x, y][[z]]$.

The same kind of arguments can be used to show that the generic fiber of the extension

$$K[x, y_1, \dots, y_n][[z]] \hookrightarrow K[x, y_1, \dots, y_n]_x[[z]]$$

has dimension $\geq n$.

Finally, using this fact and Cohen's structure theorem for complete local rings, we establish the following result. Let B be a Noetherian domain containing a field, $a \in B$ a nonzero nonunit and z an indeterminate over B . Then the generic fiber of the extension $B[[z]] \hookrightarrow B_a[[z]]$ has dimension $\geq \dim(B/aB)$. In particular, $B[[z]] \hookrightarrow B_a[[z]]$ is not TGF when $\dim(B/aB) \geq 1$.

2. Results.

Let K be a field and x, z indeterminates over K . In [4, Proposition 2.6], it is shown that the extension $K[x][[z]] \hookrightarrow K[x]_x[[z]]$ is TGF. We extend this result.

Proposition 1. *Let A be a one-dimensional domain, $a \in A$ a nonzero nonunit and x an indeterminate over A . Then the extension $A[[x]] \hookrightarrow A_a[[x]]$ is TGF.*

Proof. We adapt the proof of [4, Proposition 2.6]. Suppose that the extension is not TGF. Hence there exists a nonzero prime ideal P of $A_a[[x]]$ such that $P \cap A[[x]] = 0$. Since $x \notin P$, there exists an $f \in P$ such that $0 \neq f(0) \in A$.

Let u be the canonical map $A[[x]] \rightarrow A_a[[x]]/fA_a[[x]]$. Reducing u modulo (x) we get the canonical map $A \rightarrow A_a/f(0)A_a$ which is surjective because $A/f(0)A$ is zero-dimensional, cf. [5, Theorem 3.1]. It is well-known that $A[[x]]$ is complete in the (x) -adic topology. Moreover, by the next lemma, $A_a[[x]]/fA_a[[x]]$ is separated in the (x) -adic topology. By Cohen's theorem (see [6, Lemma on page 212] or [3, Lemma 4.2]), u is surjective. Since $fA_a[[x]] \cap A[[x]] \subseteq P \cap A[[x]] = 0$, u is an isomorphism. This is a contradiction, because a is invertible in $A_a[[x]]/fA_a[[x]]$ but non-invertible in $A[[x]]$. •

The following lemma is probably well-known, but we were unable to find a reference for it.

Lemma 2. *Let A be a domain and $0 \neq f \in A[[x]]$. Then the principal ideal $fA[[x]]$ is closed in the (x) -adic topology.*

Proof. Let $g \in \cap_{m \geq 1} (f, x^m)A[[x]]$. Clearly, we may assume that $c = f(0)$ is nonzero. Then $g = fh$ for some $h \in A_c[[x]]$, say, $h = \sum_{n \geq 0} h_n x^n$ with $h_n \in A_c$. Let $k \geq 1$. As $g \in (f, x^k)A[[x]]$, $g = fq + x^k r$ for some $q, r \in A[[x]]$. So $g = fh = fq + x^k r$, hence $f(h - q) = x^k r$. As $f(0) \neq 0$, $h - q = x^k s$ for some $s \in A_c[[x]]$. Hence $h_0, h_1, \dots, h_{k-1} \in A$. So $h \in A[[x]]$, that is, $g \in fA[[x]]$. •

Let $R \subseteq T$ be an extension of domains and $\lambda_1, \dots, \lambda_n \in zT[[z]]$, where z is an indeterminate over T . Recall that $\lambda_1, \dots, \lambda_n$ are said to be *analytically independent* over $R[[z]]$ if the $R[[z]]$ -algebra morphism

$$\theta : R[[z]][[y_1, \dots, y_n]] \rightarrow T[[z]]$$

given by $\theta(y_i) = \lambda_i$, $1 \leq i \leq n$, is injective, where y_1, \dots, y_n are indeterminates over $R[[z]]$. The next proposition is the key technical result of this note.

Proposition 3. *Let $A \subseteq A'$ be an extension of domains, $a \in A$ a nonzero nonunit and y, z indeterminates over A' . Assume there exists an element in $\lambda \in zA_a[[z]]$ which is analytically independent over $A'[[z]]$. Then the extension $B[[z]] \hookrightarrow B_a[[z]]$ is not TGF for every ring B between $A[y]$ and $A'[[y]]$.*

Proof. Let P be the kernel of the $A'_a[[z]]$ -algebra morphism

$$\theta : A'_a[[z, y]] \rightarrow A'_a[[z]]$$

given by $\theta(y) = \lambda$. Note that $0 \neq \lambda - y \in P \cap B_a[[z]]$. Also

$$(P \cap B_a[[z]]) \cap B[[z]] \subseteq P \cap A'[[z, y]] = 0$$

because λ is analytically independent over $A'[[z]]$. •

Corollary 4. *Let D be a domain and x, y, z indeterminates over D . Then the extension $B[[z]] \hookrightarrow B_x[[z]]$ is not TGF for every ring B between $D[x, y]$ and $D[[x, y]]$.*

Proof. It suffices to show that there exists an element of $zD[x]_x[[z]]$ which is analytically independent over $D[[x, z]]$, because, after that, we apply Proposition 3 for $A = D[x]$, $A' = D[[x]]$ and $a = x$.

Let K be the quotient field of D . Let $\sigma(z) \in zD[[z]]$ be an element which is algebraically independent over $K(z)$ (e.g., $\sigma(z) = \sum_{n \geq 1} z^{n!}$, cf. [2, page 277]). By [1, Theorem 2], $x\sigma(z)$ is analytically independent over $K[[x, xz]]$. So $x\sigma(z/x) \in zD[x]_x[[z]]$ is analytically independent over $D[[x, z]]$. •

The next corollary answers [4, Question 4.8].

Corollary 5. *Let K be a field and x, y, z indeterminates over K . Then the extension $K[x, y][[z]] \hookrightarrow K[x, y]_x[[z]]$ is not TGF.*

Proof. We apply Corollary 4 for $D = K$ and $B = K[x, y]$. •

Recall that the *generic fiber* of an extension of integral domains $A \subseteq B$ is the set of prime ideals of B lying over zero in A .

Remark 6. Let K be a field and x, z, y_1, \dots, y_n indeterminates over K with $n \geq 1$. The arguments employed in the proofs of Proposition 3 and Corollary 4 can be used to show that the generic fiber of the extension

$$K[x, y_1, \dots, y_n][[z]] \hookrightarrow K[x, y_1, \dots, y_n]_x[[z]]$$

has dimension $\geq n$.

Indeed, let $\sigma_1(z), \dots, \sigma_n(z) \in zK[[z]]$ be algebraically independent elements over $K(z)$. By [1, Theorem 2], the elements $\lambda_j = x\sigma_j(z/x)$, $1 \leq j \leq n$, are analytically independent over $K[[x, z]]$. Clearly, $K[[x]]_x$ is the quotient field $K((x))$ of $K[[x]]$. For $1 \leq j \leq n$, let P_j be the kernel of the $K((x))[[z]]$ -algebra morphism

$$\theta : K((x))[[z, y_1, \dots, y_n]] \rightarrow K((x))[[z, y_{j+1}, \dots, y_n]]$$

given by $\theta(y_i) = \lambda_i$ for $1 \leq i \leq j$ and $\theta(y_i) = y_i$ for $j+1 \leq i \leq n$. Then $y_j - \lambda_j \in P_j \setminus P_{j-1}$ (in fact, it is easy to see that P_j is the ideal generated by $y_1 - \lambda_1, \dots, y_j - \lambda_j$). Set $Q_j = P_j \cap K[x, y_1, \dots, y_n]_x[[z]]$, $1 \leq j \leq n$. Then

$$0 = Q_0 \subset Q_1 \subset Q_2 \subset \dots \subset Q_n$$

the inclusions being proper because $y_j - \lambda_j \in Q_j \setminus Q_{j-1}$. Since $\lambda_1, \dots, \lambda_n$ are analytically independent over $K[[x, z]]$, we get

$$0 = P_n \cap K[[x, z, y_1, \dots, y_n]] \supseteq Q_n \cap K[x, y_1, \dots, y_n][[z]].$$

Note that the elements $\sigma_1(z), \dots, \sigma_n(z)$ above can be chosen in $z\Omega[[z]]$, where Ω is the prime subfield of K . Indeed, if $\sigma_1(z), \dots, \sigma_n(z) \in z\Omega[[z]]$ are algebraically independent over $\Omega(z)$, then, by base extension, they are also algebraically independent over $K(z)$, because the canonical morphism $\Omega[[z]] \otimes_{\Omega} K \rightarrow K[[z]]$ is injective (since every Ω -vector space basis of K is linearly independent over $\Omega[[z]]$). •

The next corollary was suggested by the proof of [4, Proposition 4.9].

Corollary 7. *Let D be a domain, x, y, z indeterminates over D and $a \in D$ a nonzero nonunit such that $\cap_{n \geq 1} a^n D = 0$. Then the extension $B[[z]] \hookrightarrow B_a[[z]]$ is not TGF for every ring B between $D[x, y]$ and $D[[x, y]]$.*

Proof. It suffices to show that there exists an element of $zD[x]_a[[z]]$ which is analytically independent over $D[[x, z]]$, because, after that, we apply Proposition 3 for $A = D[x]$ and $A' = D[[x]]$.

By [7, Theorem 2.1], there exist $\sigma \in zD[[z/a]]$ which is algebraically independent over $D[[z]]$. Now we use the proof of [4, Proposition 4.9] to show that $\lambda = \sigma x$ is analytically independent over $D[[x, z]]$. Indeed, let $f(y) \in D[[x, z]][[y]]$, such that $f(\lambda) = 0$. Writing $f(y) = \sum_{l=0}^{\infty} \sum_{i+j=l} d_{ij}(z) x^i y^j$ with $d_{ij}(z) \in D[[z]]$, we get

$$0 = f(\lambda) = \sum_{l=0}^{\infty} \sum_{i+j=l} d_{ij}(z) x^i (\sigma x)^j = \sum_{l=0}^{\infty} x^l \sum_{i+j=l} d_{ij}(z) \sigma^j.$$

Hence $\sum_{i+j=l} d_{ij}(z) \sigma^j = 0$ for each l . As σ is algebraically independent over $D[[z]]$, each $d_{ij}(z) = 0$. Thus $f(y) = 0$. •

The following proposition is the main result of this note. It is a generalization of Remark 6.

Proposition 8. *Let B be a Noetherian domain containing a field, $a \in B$ a nonzero nonunit and z an indeterminate over B . Then the generic fiber of the extension $B[[z]] \hookrightarrow B_a[[z]]$ has dimension $\geq \dim(B/aB)$. So, $B[[z]] \hookrightarrow B_a[[z]]$ is not TGF when $\dim(B/aB) \geq 1$.*

Proof. Clearly, we may assume that $\dim(B/aB) \geq 1$. Let $1 \leq n \leq \dim(B/aB)$ and let M be a prime ideal of B of height $n+1$ containing a . Set $C = B_M$ and let \widehat{C} be the completion of C . By [6, Theorem 60], \widehat{C} contains a coefficient field K . Pick $b_1, \dots, b_n \in B$ such that a, b_1, \dots, b_n is a system of parameters of C . Then a, b_1, \dots, b_n is also a system of parameters of \widehat{C} . By the proof of Cohen's structure theorem for complete local rings [6, Corollary 2, page 212] (see also [3, Theorem 4.3]), the K -algebra morphism

$$\theta : K[[x, y_1, \dots, y_n]] \rightarrow \widehat{C}$$

given by $\theta(x) = a$ and $\theta(y_j) = b_j$, $1 \leq j \leq n$, is injective and finite. We may assume that θ is the inclusion map. So $x = a$ and $y_j = b_j$, $1 \leq j \leq n$.

Let Ω be the prime subfield of K . By Remark 6, there exist the elements $\lambda_1, \dots, \lambda_n \in z\Omega[x]_x[[z]]$ which are analytically independent over $K[[x, z]]$ and a chain of prime ideals of $K((x))[[z, y_1, \dots, y_n]]$

$$0 = P_0 \subset P_1 \subset P_2 \subset \dots \subset P_n$$

such that $y_j - \lambda_j \in P_j \setminus P_{j-1}$ for $1 \leq j \leq n$ and $P_n \cap K[[x, z, y_1, \dots, y_n]] = 0$. Set $D = K[[x, y_1, \dots, y_n]]$ and $P'_j = P_j \cap D_x[[z]]$, $1 \leq j \leq n$. Then

$$0 = P'_0 \subseteq P'_1 \subseteq P'_2 \subseteq \dots \subseteq P'_n.$$

Since $\widehat{C}_x[[z]]$ is a finite $D_x[[z]]$ -module, there exist a chain of prime ideals of $\widehat{C}_x[[z]]$

$$0 \subseteq Q_1 \subseteq Q_2 \subseteq \dots \subseteq Q_n$$

such that $P'_j = Q_j \cap D_x[[z]]$ for $1 \leq j \leq n$, cf. [6, Theorem 5]. Set $T_n := Q_n \cap \widehat{C}[[z]]$. Then

$$T_n \cap D[[z]] = P'_n \cap D[[z]] = P_n \cap D[[z]] = 0.$$

As $\widehat{C}[[z]]$ is a finite $D[[z]]$ -module, it follows that T_n is a minimal prime ideal of $\widehat{C}[[z]]$, cf. [6, Theorem 5]. Since $\widehat{C}[[z]]$ is a flat $B[[z]]$ -module, we get $T_n \cap B[[z]] = 0$, cf. [6, Theorem 4]. Set $N_j = Q_j \cap B_x[[z]]$, $1 \leq j \leq n$. Note that $N_n \cap B[[z]] \subseteq Q_n \cap \widehat{C}[[z]] = 0$, so each N_j is in the generic fiber of $B[[z]] \hookrightarrow B_x[[z]]$. The inclusions

$$0 = N_0 \subset N_1 \subset N_2 \subset \dots \subset N_n$$

are proper because $y_j - \lambda_j \in N_j \setminus N_{j-1}$ for $1 \leq j \leq n$. Thus the generic fiber of $B[[z]] \hookrightarrow B_x[[z]]$ has dimension $\geq n$. •

Remark 9. The assertion of Proposition 8 does not hold for non-Noetherian domains. For example, let V be a rank-two valuation domain containing a field and let a be a nonzero element of the height-one prime ideal of V . Then V_a is the quotient field of V , so the extension $V[[z]] \hookrightarrow V_a[[z]]$ is TGF, but $\dim(V/aV) = 1$.

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